A FEW THOUGHTS ON THE PUTNAM

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Introduction

The William Lowell Putnam Mathematical Competition, colloquially known as the "Putnam", is a prestigious annual event open to all regular undergraduate students enrolled at universities and colleges in the U.S. and Canada. On the first Saturday in December, students who have registered for the competition gather for two three-hour sessions, separated by a two-hour lunch break. In each session, there are six problems; the students work on these individually, with no communication even between the members of a team.

The Putnam is held almost simultaneously on a large number of university and college campuses across the two countries (as well as at the Budapest Semesters in Mathematics program in Hungary). Starting times are staggered by time zone to minimize the risk of unauthorized information being transmitted, say, from the East to the West Coast. Because of the large number of venues (the 2013 Putnam had contestants from 557 institutions), there is no practical way for any corrections to be disseminated, and local supervisors are instructed not to answer any questions or offer any comments. Thus the exam must be worded very carefully and accurately, with as much consideration as possible for the many contestants for whom English is not a native language, in particular foreign students and students from the Canadian province of Quebec. The latter may, and sometimes do, answer the questions in French.

Substantial cash prizes (in 2013, the top prize was $25,000) are awarded to the mathematics departments of the five top teams in the competition, as determined by the, perhaps surprising, method explained below. There are also generous cash awards for the top 25 individual scores, as well as for the members of the top teams (who often collect individual awards as well). One of the top five individuals (who are known as the Putnam Fellows) receives a fellowship for graduate study at Harvard. Since 1992, there has been a special award available for "a woman whose performance on the Competition has been designated particularly meritorious". A list of the top 500 or so individual participants with a very rough indication of their rankings (for instance, in 2013 ranks 26 through 76.5 merited Honorable Mention) is published; it is presumed that graduate schools in mathematics will consider success on the Putnam to be a good indicator of likely success in their programs. Certainly, over the years, a fair number of the top contestants have developed into world-class mathematicians (and physicists), including winners of the Fields Medal and the Nobel Prize. For examples, see the section "A Putnam Who's Who" in [Gallian].

Past problems and solutions from the Putnam can be found in the collections [Gleason], [Alexanderson], and [Kedlaya]. Problems and solutions are also published annually in the *American Mathematical Monthly* and in *Mathematics Magazine*. On the Web, Kedlaya's Putnam archive [Archive] contains both problems and unofficial but excellent solutions.

In [Reznick], the author offers a vivid description, from personal experience, of the none-too-systematic process by which problems for the Putnam are composed and selected; the description is followed first by commentary by Loren Larson and then by a wider discussion. This is also the general topic on which it was suggested that I speak at the conference in Barranquilla. Although I am unlikely to match Reznick's eloquent insights unless I paraphrase them, I will try to provide a reflective update here, drawing on my own experiences both as a problem setter for the Putnam and, more recently, as Larson's successor in the role of liaison to the committee of problem setters.

Evolution and subject matter of the Putnam

The early history of the Putnam, including the idea by its namesake that led to the founding (and funding!) of the competition, is traced in the short article [Birkhoff], which is reprinted in [Gleason]. At its inception the contest seems to have been intended more as a test of thorough mastery of standard material than as a challenge to the ingenuity, creativity, and insight of the participants. For example, the fifth problem on the first Putnam (which was held in 1938, and taken by 163 students) asked for the evaluation of the limits

$$\lim_{n\to\infty} \frac{n^2}{e^n} \quad \text{and} \quad \lim_{x\to 0} \frac{1}{x} \int_0^x (1+\sin 2t)^{1/t}\, dt\,,$$

which requires little more than l'Hôpital's Rule, and which leaves the definite impression that the first limit is there to give a hint on how to approach the second one. After a hiatus from 1943 through 1945 due to the Second World War, the Putnam resumed with a rather different flavor. [Birkhoff] explains that the setting of the problems, originally done by the mathematics department whose team had won the previous year (and which was then ineligible to win for one year), was turned over to a "special committee". That first committee consisted of Pólya, Radó, and Kaplansky (who had been the first recipient of the annual Putnam fellowship). To give an example of the change in emphasis, here is an elegant problem from the 1947 Putnam:

> a, b, c, d are distinct integers such that
> $$(x-a)(x-b)(x-c)(x-d)-4=0$$
> has an integral root r. Show that $4r = a+b+c+d$.

Had this problem been new last year, it would have seemed equally appropriate for the more than 4000 participants taking the 2013 Putnam.

Here is the current official "description" of the Putnam Competition, as found on the Putnam website [Web]:

> The examination will be constructed to test originality as well as technical competence. It is expected that the contestant will be familiar with the formal theories embodied in undergraduate mathematics. It is assumed that such training, designed for mathematics and physical science majors, will include somewhat more sophisticated mathematical concepts than is the case in minimal courses. Thus the differential equations course is presumed to include some references to qualitative existence theorems and subtleties beyond the routine solution devices. Questions will be included that cut across the bounds of various disciplines, and self-contained questions that do not fit into any of the usual categories may be included. It will be assumed that the contestant has acquired a familiarity with the body of mathematical lore commonly discussed in mathematics clubs or in courses with such titles as 'survey of the foundations of mathematics'. It is also expected that the self-contained questions involving elementary concepts from group theory, set theory, graph theory, lattice theory, number theory, and cardinal arithmetic will not be entirely foreign to the contestant's experience.

Despite the presence of this description, there is clearly a good deal of room for uncertainty and disagreement about the desired or acceptable scope of Putnam problems. Even among the problem setters, different people have different ideas about the range of "undergraduate mathematics"; the reference to "physical science majors" may suggest an emphasis on calculus, linear algebra, and differential equations (perhaps extended to real analysis by the words "formal theories"), but are Fourier series fair game? Contour integrals? Generating functions? The Cayley-Hamilton theorem? Galois theory? Given that there are many respectable undergraduate institutions (including my own) where the requirements for a mathematics major no longer include a specific course in differential equations, perhaps that reference in the description should be updated. Certainly I would be surprised these days to see a Putnam problem comparable to the following, which occurred in the 1957 competition:

> The curve $y = f(x)$ passes through the origin with a slope of 1. It satisfies the differential equation $(x^2+9)y'' + (x^2+4)y = 0$. Show that it crosses the x-axis between $x = \frac{3}{2}\pi$ and $x = \sqrt{\frac{63}{53}}\,\pi$.

(By the way, no calculators or other technology have ever been allowed on the Putnam.) To solve this problem, contestants were apparently expected to use the Sturm Comparison Theorem, comparing the given differential equation to $9\,y'' + 4\,y = 0$. This problem was, in fact, one of several singled out for reproach in a 1963 article by Mordell (reprinted, together with a rejoinder by L. M. Kelly, in [Gleason]) containing extensive criticism of the Putnam. Mordell's comment about a different problem: "The question places a great premium on knowledge far beyond what most undergraduates know, for I cannot believe that a student is likely to find a proof during the examination" seems equally applicable to this one.

The current situation is complicated by the fact that it is fairly common for strong students at top universities to be attending graduate courses even while they are technically still undergraduates. Thus it seems unfair to other contestants if problems (some would say: any but the hardest problems) are significantly easier to solve when graduate-level techniques are used.

Problem setters and problem formulation

Just as when it boasted Pólya, Radó, and Kaplansky, the Questions Committee, which composes and assembles the Putnam, still consists of three problem setters; there is also a liaison to the directors of the competition. The problem setters serve overlapping three-year terms on the committee, so that each year one problem setter is replaced. On occasion, problem setters have returned to the committee, after an interval, for a second three-year term. The liaison, whose role is largely administrative, is expected to contribute a sense of long-term perspective and continuity to the process of creating the exam.

Because the problem setters are typically in very different parts of the subcontinent and most often have full-time academic positions, in practice the Questions Committee can only meet once, for a day and a half to two days, on some weekend. In preparation for this meeting, proposed problems and solutions are exchanged and commented on, but the first part of the meeting is inevitably needed to make final choices of problems and their order on the exam. Once the problems are chosen, naturally their solutions will be further scrutinized and streamlined. However, at the meeting, more time is spent on reviewing and fine-tuning the exact wording of the problems. It is not uncommon for the committee to spend an hour or more on the formulation of one problem, and even the need for, or redundancy of, a single comma can become a topic of extended (but usually friendly) dispute. To make the problems unambiguously clear to as many contestants as possible, sometimes the definition of a relatively standard concept is included, or a seemingly redundant clarification is added. Despite all this care, there have been cases in which the final wording turned out to be susceptible to misinterpretation.

For example, the first problem for the afternoon session of the 2010 Putnam was stated as follows:

> B1. Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that
> $$a_1^m + a_2^m + a_3^m + \cdots = m$$
> for every positive integer m?

It may seem clear what was intended here, and everyone on the committee was satisfied with the formulation. However, after the exam, complaints were received on behalf of contestants who had interpreted the problem as

> B1*. Is it true that for every positive integer m, there is an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that
> $$a_1^m + a_2^m + a_3^m + \cdots = m\,?$$

It was even argued that this *had* to be the intention, because otherwise the problem would presumably have been stated as

> B1**. Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that for every positive integer m,
> $$a_1^m + a_2^m + a_3^m + \cdots = m\,?$$

In hindsight, B1** would have been a better formulation, but I still have only limited sympathy for contestants who seriously thought that B1* was intended – how could a "problem" with such an obvious solution (take, say, $a_1 = m^{1/m}$ and $a_2 = a_3 = \cdots = 0$) ever occur on the Putnam?

Problem selection

First of all, the committee has to work with the problems provided by the problem setters (and occasionally by the liaison). For at least the last twenty years or so, the guideline has been that each setter should provide at least ten candidate problems. For security reasons, it is best if these are newly created for the occasion; sometimes setters have accepted problems for consideration from sources they trust (who undertake not to share such problems with anyone else), but this has occasionally led to awkward situations. For example, it was noticed after last year's exam that one of the problems had appeared in *Mathematics Magazine* in 1986. Fortunately, that was well before the current contestants were born, and with luck none of them had been exposed to the problem. It has even been suggested to expand the committee by one additional member, specifically to take on responsibility for checking that none of the problems are known. Of course, the liaison would already be expected to notice if a problem was too similar to one that had appeared previously on the Putnam.

Before the meeting, committee members are asked to try solving each others' candidate problems before reading the solutions (often the problems are distributed earlier than the solutions, to reduce temptation). They then rate the problems, both as to difficulty level and for "suitability" for the Putnam. Difficulty is easy to define: A problem has difficulty level d if its difficulty would suggest placing it as the d-th problem in one of the two sessions of the Putnam. (Within each session, the problems are arranged in what the committee perceives, not always correctly, to be increasing level of difficulty. This seems not to have been the case in the early years of the Putnam.) As for suitability, after ruling out some candidate problems because they turn out to be known (this year, one such was found on Wolfram MathWorld™) and others because they have difficulty level 0 or 7, the committee members are left with value judgments such as the ones described in [Reznick]: Is the problem interesting? Does its solution provide a sense of satisfaction? Especially for the earlier problems, which will be attempted by thousands of contestants: How time-consuming will it be to grade this? For example, one redeeming feature of problem A1 from 2011 (see [Archive]) was its answer, 10053. Even though it followed relatively quickly from the right insight, this answer seemed unlikely to be found by anyone who did not have a valid solution.

Once the candidate problems are rated, assembling twelve of the better ones into a viable exam can range, depending on the year, from a breeze to a nightmare. Is there a suitable range of difficulty? In particular, are there problems that are "easy" enough that a positive median score can be expected? If that seems like a surprising question, here's the bad news: Despite the committee's best intentions, in some years more than half the contestants have not scored *any* points out of the possible 120. This is a bit demoralizing for all concerned, especially because the contestants are typically among the better mathematics students at their institutions. They are aware that the Putnam is supposed to be challenging, but not that it will be *that* challenging. On the other end of the scale, are there problems that are challenging enough so that almost no one will approach closely that mythical score of 120? After all, it would be most unpleasant to have to resort to hair-splitting to pick the top five out of a larger number of near-perfect efforts. Another question: Is there a reasonable distribution of problems over different areas of undergraduate mathematics, and over topics within those areas? For example, at one stage of the preparation of the 2011 exam, there seemed to be a risk that at least half the problems would ask for the proof of an estimate (inequality) – in the end this was cut down to a "mere" three estimations.

By the end of the meeting, the committee has typically spent so much time considering the problems that were chosen, and therefore has become so familiar with how they "work", that a good deal of doubt starts to creep in. In fact, the same committee member may, within the space of half an hour or so, express reservations because the exam, as fine-tuned, might now be too easy – and then express concern because it might be too difficult, after all. The best remedy for this quandary seems to be to create some distance by flying back to one's home institution!

How to create problems?

I wish I knew. [Reznick] offers good suggestions, and then ends up admitting that the question "How do you sit down and create" is "very difficult and personal" and perhaps not answerable. In my own experience, when I was searching for just about any problem at all (for years, I was trying to supply two new problems a week for our departmental newsletter) I would often do random little mathematical experiments until I noticed something interesting or I became curious about something, then see whether I could create a problem out of what I had just noticed or whether I could answer my own question. Given that the main requirement for these departmental "Problems of the Week" was a somewhat novel statement that might appeal to our students and that some of them would have a fighting chance of solving, perhaps a third of these random attempts would pan out. For Putnam problems, which have much more rigid constraints (for example, two pages of routine calculation as part of a solution is really not acceptable), the success rate was much lower. Meanwhile, a few of the Putnam problems that I am happiest to have "produced" seem to have occurred to me without any warning or even conscious thought.

Grading

The examination papers are sent promptly to the Director of the Competition, Leonard Klosinski, at Santa Clara University, and a four-day preliminary grading "marathon" (two days for each of the three-hour problem sessions) is held there later in the month. The grading is very stringent; rigorous arguments are expected, and to get even a minimal amount of partial credit for a problem, a contestant must show significant progress. Meanwhile, it is not unheard of for Putnam graders to be confronted with student solutions asserting that some claim used without further proof in the solution follows directly from, say, the "well-known PQR Theorem", when none of the graders in the room feels confident even of the statement of that theorem, let alone of how it might apply to the situation at hand. In the majority of such cases, in the end no credit can be given, but this is only decided after enough time is spent to make sure that the solution is actually wrong or hopelessly incomplete – which may involve consulting references.

Once the preliminary grading is over there is considerable cross-checking, including a complete regrade of the work of the top 200 or so participants to ensure accuracy and consistency. The team rank for each institution is then calculated on the basis of the *average* (or the sum) of the individual *ranks* of three contestants from that institution. Those three "team members" must be designated in advance of the competition; the intent of this rule seems to be to lessen somewhat the advantage held by schools such as M.I.T. and Harvard that have a large number of very strong contestants every year – but that may have some trouble predicting which of their many entrants will get the best scores. As pointed out in [Gallian], one result of this method is that it is even more important for an institution not to designate a team member who may get a very low score than to designate the ones who will score highest. For example, in 2013 a score of 30 corresponded to a rank of 266, a score of 20 to a rank of 597, and a score of 10 to a rank of 1324. Thus a team whose members scored 30, 30, 10 would have been in a worse position than one with scores of 20, 20, 20. Typically, the final results of the competition, including both team and individual rankings, are distributed to the local supervisors in March.

Incidentally, by far the most entertaining student answer from my own grading experience was for problem B4 from the 1994 Putnam:

> For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where
>
> $$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
>
> Show that $\lim_{n \to \infty} d_n = \infty$.

In response, one contestant wrote approximately the following:

> There was once a matrix whose powers'
> Gcd opened like flowers;
> They went through the roof,
> But I don't have a proof,
> And now it's the end of three hours.

Alas, under the grading criteria it was impossible to reward this particular bit of creativity.

Challenges and conclusion

As alluded to earlier, there can be a wide disparity in knowledge beyond "the" undergraduate curriculum among participants of comparable age and talent, due to circumstances beyond their control. There have been philosophical differences between problem setters regarding the best way to respond to this situation. For instance, is an "ideal" Putnam problem one that not only admits a relatively short and elegant solution, but also gives contestants a preview of an idea that may be significant in their later mathematical development – while potentially giving an important advantage to a select few who may have been introduced to that idea? Or is it better to stay with problems that, as far as the problem setters know, are "dead ends" where only *ad hoc* ingenuity and fairly standard material are useful? Of course, the boundary between these two categories is a bit hazy. For example, here is problem A6 from the 1995 Putnam:

> Suppose that each of n people writes down the numbers $1, 2, 3$ in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

If memory serves, no one on the committee was aware, when this problem was chosen, that it could be solved using the Local Central Limit Theorem applied to a random walk on a triangular lattice. My own solution, which was published in the "official" account of the Putnam (see [Klosinski]), was distinctly *ad hoc* and unlike two further solutions that subsequently appeared in the superlative collection [Kedlaya]. On the other hand, it appears that this problem was solved by only two contestants; I have no idea whether either of them applied the Local Central Limit Theorem, or was even in a position to do so.

Despite the shifts in subject matter mentioned earlier, much of the "core" material for the Putnam remains the same over the years, which presumably means that it will become increasingly difficult to come up with truly fresh problems, especially elementary ones. This is almost certainly true for any individual problem setter, and even with constant "new blood" on the Questions Committee it may eventually turn out, in a variation on Lagrange's famous pessimistic phrase (see [Stillwell]), that "the mine is already too broad".

Over the years, the number of students taking the Putnam has risen irregularly but quite significantly, outpacing the overall student population; it first reached 1000 in 1961, 2000 in 1973, 3000 in 2002, and 4000 in 2009. (For details, see Table 1 in [Gallian].) Last year's attendance of 4113 should be considered artificially low, because a substantial part of the south-central U.S. was affected by an unusually early and destructive ice storm which made travel hazardous and forced a number of institutions to cancel their sessions of the Putnam. While the growth in student interest is welcome, it first made the "marathon" grading sessions more onerous and eventually led to a significant increase in the number of graders required. It seems astonishing to read, in an article by L. E. Bush reprinted in [Gleason], that Putnam Competitions 9 through 19 were completely graded by one single grader (who must have had amazing stamina, since the nineteenth competition had 506 participants). Naturally, the potential for inconsistency increases with the number of graders for a single problem. Perhaps more importantly, the gap between the best and the least prepared undergraduates taking the exam seems to be growing over time. There have been suggestions that as a result of the effort to make at least one or two problems on the Putnam feasible for most participants without being completely trivial, the strongest participants are being made to do what for them is a significant amount of "busywork", which cuts into the time they have to work on the more "interesting" problems. However, so far the proposals I have heard that intend to remedy this, such as having a "qualifying" round of problems, perhaps to be taken electronically, that would lead to an especially demanding "final" round, have their own substantial drawbacks. In any case, it can be argued that any sort of timed competition in which one is expected to have "original" ideas will sometimes lead to capricious and/or discouraging results. It can also be argued that many, and I hope most, Putnam participants ultimately derive considerable satisfaction from the challenge of the activity, and the interest and beauty of some of the problems, with which they choose to wrestle on the first Saturday in December.

Acknowledgements and disclaimer: It should surely be mentioned that for many years now, the Putnam has benefited greatly from the tireless organizational efforts of Jerry Alexanderson and Leonard Klosinski and their supporting staff at the University of Santa Clara. On a personal note, I would especially like to thank Loren Larson, who spent many years in the role of liaison and whose excellent work I can only hope to emulate, for very helpful conversations in connection with my writing this article, and in particular for drawing my attention to [Reznick]. The opinions expressed above are my own and should not be taken as official pronouncements by either the Putnam Competition or the Mathematical Association of America (which administers the Putnam).

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