

Predicting the Future: Four Classic Conjectures of Mathematics

by Alexander Soifer

Abstract

By “classic” I mean here problems that are easy to understand but not easy if at all possible to solve. In mid 1960s Andrej Kolmogorov at the award presentation lecture of the Moscow University Mathematical Olympiad, which I attended, said, “Perhaps, the only way to get a proof of Fermat’s Last Theorem is to offer it at this Olympiad.” Perhaps, this is the only way to recover Fermat’s original proof!

In this talk I will show how beautiful ideas extracted from mathematical research can give birth to new problems for Mathematical Olympiads.

Conversely, Mathematical Olympiads feed research. I will show some open problems of mathematics that were born while I was creating problems for the Colorado Mathematical Olympiad.

PART Naught: OVERTURE

What is creating conjectures about?

Surely, it is the art of predicting the future.

Niels Bohr took such predicting jokingly:

Predicting is very difficult, especially the future.

Albert Einstein ó nonchalantly:

I never think of the future ó it comes soon enough.

In conjecturing, we use the power of intuition to envision the result without being able to prove it.

In 1989 after Paul Erdős lecture here at the University of Colorado, Professor of Mathematics Gene Abrams was unimpressed, "What is a big deal about posing problems? Proving results is much more important." "Without someone posing problems, and moreover predicting results by conjectures, as Erdős has done, there will be nothing to prove," I replied. When we commence research, we do not know what is true, and let our intuition lead the way. We have to rely on insight or good luck in choosing a conjecture to prove. And if we choose a conjecture that is not true, it would take a very long time to prove it, a very-very long time!

I oppose discrimination of young high school and college mathematicians based on their youth and inexperience, as Ronald Reagan once remarked. Thus, I will present here unsolved problems, unproven conjectures that are waiting for their conquerors. There is plenty of contemporary mathematics, hidden behind an elaborate maze of definitions, and sometimes consisting of merely juggling with them. This kind of juggling does not interest me much. I prefer classic problems. By "classic" I do not necessarily mean problems that are centuries old, but rather problems that are easy to understand by anyone, including a middle school student or a layman, but tantalizingly difficult if at all possible to solve. There are additional conditions on admission of a problem into "classic" category: an aesthetic appeal of a problem is essential, as are expectations that the result will defy our intuition.

In this essay I will demonstrate interaction between Olympiad and research problems, and present "live" fragments of mathematics, centered on predicting the future by formulating classic conjectures. I am offering you a journey on a mathematical train of thought through problems, conjectures, and results. I hope you will enjoy the ride!

CONJECTURE I. Squares in a Square (Erdős 1932)¹

In 1994 Paul Erdős shared with me one of his oldest conjectures.

We are in Budapest, Hungary, rolled back 82 years. In 1932, the 19-year old Paul Erdős poses the following problem. Inscribe in a unit square r squares, which have no interior points in common. Denote by $f(r)$ the maximum of the sum of side lengths of the r squares. (We allow side lengths to be zero.) The problem is to evaluate the function $f(r)$:

Open Problem 1. For every positive integer r find the value of $f(r)$.

In fact, this formulation came about later, when Paul and I renewed efforts to settle the problem. Originally Paul formulated the following narrow but surprisingly difficult conjecture. When Paul shared the conjecture with me, he offered a \$50 price for its first proof or disproof.

Fifty Dollar Squares in a Square Conjecture 2 (Paul Erdős, 1932). For any positive integer k ,

$$f(k^2 + 1) = k.$$

The conjecture is still open today, in the year 2014, waiting, as Paul Erdős used to say, "for stronger arms, or, perhaps, brains" to be settled. However, Paul and I reached a progress in a broader

¹ This chapter is based on sections E14 and E15 of the author's book [S6].

problem of describing the function $f(r)$. First of all we observed the following lower and upper bounds for $f(r)$. Symbol $\lfloor x \rfloor$ as usual denotes the largest integer that is not greater than x .

Result 3 (P. Erdős and A. Soifer, 1995, [ES1]). The following inequality is true for any positive integer r :

$$\lfloor \sqrt{r} \rfloor \leq f(r) \leq \sqrt{r}.$$

Proof.

1. *The Upper Bound.* The celebrated Cauchy Inequality states that

$$\left(\sum_{i=1}^r a_i b_i \right)^2 \leq \left(\sum_{i=1}^r a_i^2 \right) \left(\sum_{i=1}^r b_i^2 \right).$$

Setting $b_i = 1$ for every $i = 1, 2, \dots, r$ we get

$$\left(\sum_{i=1}^r a_i \right)^2 \leq \left(\sum_{i=1}^r a_i^2 \right) r. \quad (*)$$

Let r squares of side lengths $a_i, i = 1, 2, \dots, r$ with no interior points in common be placed in a unit square. Then the combined area of the r squares does not exceed the area of the unit square:

$\sum_{i=1}^r a_i^2 \leq 1$, and we get from the inequality (*) above the required upper bound:

$$f(r) = \sum_{i=1}^r a_i \leq \sqrt{r}.$$

2. *The Lower Bound.* Surely, the function $f(r)$ is non-decreasing, therefore, $r \geq \lfloor \sqrt{r} \rfloor^2$ implies $f(r) \geq f(\lfloor \sqrt{r} \rfloor^2)$.

Now let us partition the unit square into $\lfloor \sqrt{r} \rfloor^2$ congruent squares, each of the side length $\frac{1}{\lfloor \sqrt{r} \rfloor}$, and calculate the sum of side lengths of these $\lfloor \sqrt{r} \rfloor^2$ squares: we get $\frac{1}{\lfloor \sqrt{r} \rfloor} \times \lfloor \sqrt{r} \rfloor^2 = \lfloor \sqrt{r} \rfloor$.

Observe that this partition and the calculation demonstrate the inequality $f(\lfloor \sqrt{r} \rfloor^2) \geq \lfloor \sqrt{r} \rfloor$. By combining the two inequalities of this and the preceding paragraphs, we get the required lower bound:

$$f(r) \geq f(\lfloor \sqrt{r} \rfloor^2) \geq \lfloor \sqrt{r} \rfloor.$$

Result 3 has the following consequence:

Corollary 4 If $r = k^2$ for a positive integer k , then we get the equality $f(r) = k$.

Corollary 4 allows us to see the Erdős Fifty Dollar Squares in a Square Conjecture in a slightly different light:

Fifty Dollar Squares in a Square Conjecture 2, Second Version (P. Erdős). At perfect square numbers $r = k^2$ (k is an integer) the function $f(r)$ does not increase:

$$f(k^2 + 1) = f(k^2).$$

Paul Erdős and I were able to prove that, surprisingly, the function $f(r)$ is strictly increasing everywhere else. But to prove that we needed to find a much sharper lower bound for $f(r)$.

Theorem 6 (P. Erdős and A. Soifer, 1995, [ES1]). Any positive integer r can be presented in a form $r = k^2 + m$, where $0 \leq m \leq 2k$. Accordingly, the following inequalities hold:

(A) If $m = 2t + 1$, where $0 \leq t < k$, then $f(r) \geq k + \frac{t}{k}$;

(B) If $m = 2t$, where $0 \leq t \leq k$, then $f(r) \geq k + \frac{t}{k+1}$.

Proof. Given a positive integer r , we can present it in a form $r = k^2 + m$, where $0 \leq m \leq 2k$. Indeed, it suffices to choose $k = \lfloor \sqrt{r} \rfloor$. If r is a perfect square, $r = k^2$, then $m = t = 0$, and Corollary 4 provides the exact value $f(r) = k$, which is a part of the required inequality (B). We can assume now that r is *not* a perfect square, i.e., $m \neq 0$. The parity of m dictates two cases.

(A). $m = 2t + 1$ and $0 \leq t < k$. Let us first partition the unit square into k^2 congruent squares; we get a $k \times k$ square grid, call it G ; and then replace a $t \times t$ subgrid of the grid G with a $(t + 1) \times (t + 1)$ square grid of the same total size as the removed subgrid (Figure 1).

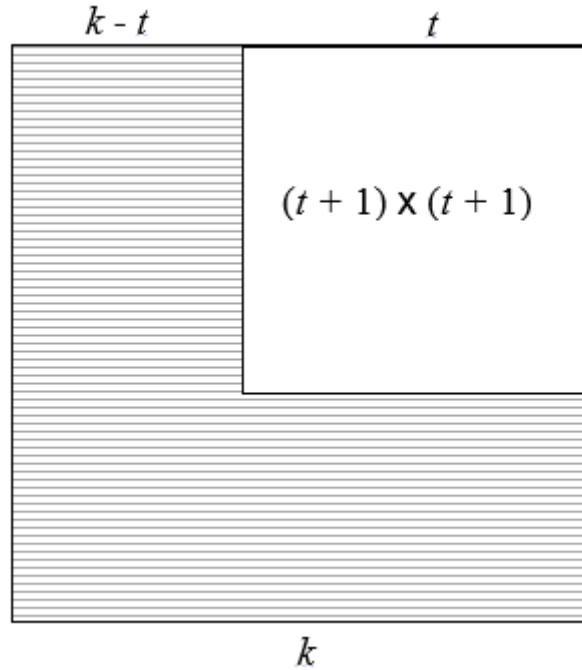


Figure 1

We end up with a partition of the unit square into $k^2 - t^2 + (t + 1)^2 = k^2 + 2t + 1$ little squares, some of which [the original ones] have side length $\frac{1}{k}$, and others [the squares of the inserted $(t + 1) \times (t + 1)$ square grid] of the side length $\frac{t}{k(t + 1)}$. Let us calculate the sum of side lengths of all these $k^2 + 2t + 1$ little squares, we get:

$$\frac{1}{k}k^2 - \frac{1}{k}t^2 + \frac{t}{k(t + 1)}(t + 1)^2 = k + \frac{t}{k}.$$

This partition and the calculation deliver the following lower bound for $f(r)$:

$$f(r) \geq k + \frac{t}{k}.$$

(B). $m = 2t$ and $0 < t \leq k$. We first partition the unit square into $(k + 1)^2$ congruent squares; we get a $(k + 1) \times (k + 1)$ square grid, call it G ; and then replace a $(k \acute{o} t + 1) \times (k \acute{o} t + 1)$ subgrid of the grid G with a $(k \acute{o} t) \times (k \acute{o} t)$ square grid of the same total size as the removed subgrid (Figure 2).

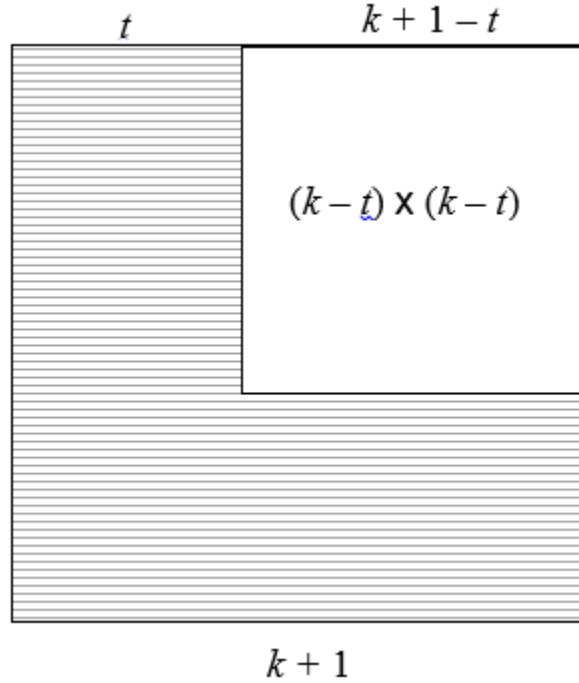


Figure 2

We end up with a partition of the unit square into $(k+1)^2 - (k-t+1)^2 + (k-t)^2 = k^2 + 2t$ little squares, some of which [the original ones] have side length $\frac{1}{k+1}$, and others [the squares of the inserted $(k-t) \times (k-t)$ square grid] of the side length $\frac{k-t+1}{(k+1)(k-t)}$. Let us calculate the sum of side lengths of all these $k^2 + 2t$ little squares, we get:

$$\frac{1}{k+1}(k+1)^2 - \frac{1}{(k+1)(k-t+1)}(k-t+1)^2 + \frac{k-t+1}{(k+1)(k-t)}(k-t)^2 = k + \frac{t}{k+1}.$$

This partition and calculation deliver the following lower bound for $f(r)$:

$$f(r) \geq k + \frac{t}{k+1}.$$

Done!

Result 7 (P. Erdős and A. Soifer, 1995, [ES1]). The function $f(r)$ is strictly increasing everywhere except possibly at perfect square points, i.e., if $r \neq k^2$ for an integer k , then $f(r+1) > f(r)$.

Proof. Once again parity of m and Theorem 6 dictate two cases.

(A). $m = 2t + 1$ and $0 \leq t < k$. In this case $t + 1 \leq k$, and by substituting $t + 1$ for t in the lower bound found in part 2 of Result 3, we get:

$$f(k^2 + 2t + 2) \geq k + \frac{t+1}{k+1}.$$

This inequality and Result 1 deliver the necessary chain of inequalities:

$$f(r) = f(k^2 + 2t + 1) \leq \sqrt{k^2 + 2t + 1} < k + \frac{t+1}{k+1} \leq f(k^2 + 2t + 2) = f(r+1).$$

(B). $m = 2t$ and $0 < t \leq k$. By using Result 1 and the lower bound found in part 1 of Result 3 above, we get the necessary chain of inequalities:

$$f(r) = f(k^2 + 2t) \leq \sqrt{k^2 + 2t} < k + \frac{t}{k} \leq f(k^2 + 2t + 1) = f(r+1).$$

Result 7 is proven.

Note: In the proof above I omitted a demonstration of two inequalities: $\sqrt{k^2 + 2t + 1} < k + \frac{t+1}{k+1}$ and $\sqrt{k^2 + 2t} < k + \frac{t}{k}$. I hope their verification would be a welcome exercise in secondary algebra for the reader.

Paul Erdős and I believed that the lower bounds in Theorem 6 were quite good, and conjectured that they just may be the best possible:

Conjecture 8 (P. Erdős and A. Soifer, 1995, [ES1]). Any positive integer r can be presented in a form $r = k^2 + m$, where $0 \leq m \leq 2k$. Accordingly, we conjecture the following equalities:

(A) If $m = 2t + 1$, where $0 \leq t < k$, then $f(r) = k + \frac{t}{k}$;

(B) If $m = 2t$, where $0 \leq t \leq k$, then $f(r) = k + \frac{t}{k+1}$.

We also observed that our examples in Theorem 6 completely tiled the unit square, and thus posed the following open problem:

Open Problem 9 (P. Erdős and A. Soifer, 1995, [ES1]). Is it true that for any positive integer r , the value of $f(r)$ can be attained by a set of r squares that form a complete tiling of the unit square by themselves or with an addition of at most *one* extra square?

As I thought about Paul Erdős's problem, it appeared natural for me to pose a dual problem, and thus give birth to the *New Squares in a Square Problem*.

Let \square stand for a square shape and $r > 1$ a positive integer. Denote by $S(\square, r)$ the smallest area of a square Q such that any r squares whose areas add up to at most 1, can be packed in Q (i.e., embedded in Q with no interior points in common).

In 1997 I offered this conjecture for small values of r at the 14th Colorado Mathematical Olympiad.

14.5. Squares in A Square (A. Soifer, [S6]).

- (A) Prove that any two squares whose areas add up to 1 can be inscribed with no interior points in common in a square of area 2.
- (B) Prove that any four squares whose areas add up to 1 can be inscribed with no interior points in common in a square of area 2.
- (C) Prove that any five squares whose areas add up to 1 can be inscribed with no interior points in common in a square of area 2.

This Olympiad assertion shows that for any r in the range $2 \leq r \leq 5$, $S(\cdot, r) = 2$. I then formulated the following conjecture:

New Squares in a Square Conjecture 10 (A. Soifer, 1996, [S1]). For any positive integer $r > 1$, $S(\cdot, r) = 2$, or to simplify notations, $S(\cdot) = 2$.

Two years have passed since I created this conjecture. In May 1997, I was in Lincoln, Nebraska grading papers of the USA Mathematical Olympiad, together with other members of USA Mathematics Olympiad Subcommittee. During a break, I put the New Squares in a Square Conjecture on the board. Later the same day Richard Stong told me "I proved your conjecture." Indeed, he did! Richard devised a simple "greedy" algorithm and a nice, clever proof that his algorithm works,² and thus New Squares in a Square Conjecture became a theorem, which I happily published in *Geombinatorics*:

Theorem 11 (R. Stong, 1997, [St]). Any finite set of squares of the combined area 1 can be packed in a square of area 2.

Later I discovered that Conjecture 10 and Theorem 11 were not new, and although Stong's proof was better, he was preceded by 30 years by J. W. Moon and Leo Moser of Edmonton, Alberta, Canada [MoM]. Ecclesiastes (1:9-14 NIV) comes to mind:

What has been will be again, what has been done will be done again; there is nothing new under the sun.

On a positive side, I brought a new excitement and new players to the problem. Moreover, I was already riding toward the next station on my train of thoughts, the one, it seems, no one has traveled before. I conjectured [S8] that the identical result was true for circular discs (I will use here the word "disc" to mean a circular disc). This 1998 conjecture is still open today, 16 years later:

Discs in a Disc Conjecture 12 (A. Soifer, [S2]). Any finite set of circular discs of combined area 1 can be packed in a circular disc of area 2: $S(O) = 2$.

What about triangles? In working with similar to each other triangles we encounter issues that had not existed for circular discs – limitations on the way "clones" are embedded. We can limit embedding to translations, and thus define a function $S_T(\Delta)$. Or we can place no limitations on embedding at all and end up with our original $S(\Delta)$. Of course, $S(\Delta) \leq S_T(\Delta)$. It was not at all

² Space considerations prevent me from including this proof here, but I recommend everyone to read it in *Geombinatorics*.

obvious whether these two values are equal. In 1995 T. J. Richardson calculated the easier of the two values.

Packing Triangles Theorem 13 (T. J. Richardson, 1995 [Ri]). Any finite set of similar to each other triangles of combined area 1 can be packed in a similar to them triangle of area 2, i.e., in my notations $S(\Delta) \leq 2$.

In 1999 I pointed out the difference in triangular embedding case and posed these triangular problems in *Geombinatorics* [S4]. In 2003 the Polish geometer Janusz Januszewski has improved Richardson's result on the pages of *Geombinatorics*:

Theorem 14 (J. Januszewski [J1]) $S(\Delta) = 2$ if and only if the triangle Δ is equilateral.

On January 27, 2009, Januszewski informed me that he calculated the harder value $S_T(\Delta)$ that I asked for in 1999 [J2]:

Packing Triangles by Translations Theorem 15 (J. Januszewski, 2009, [J2]). For any triangle Δ , $S_T(\Delta) = 2$, i.e., any finite set of similar to each other triangles of combined area 1 can be packed in a similar to them triangle of area 2 *by translations alone*.

Let us roll the time back 15 years. By 1998 I felt it was time to generalize these observations to include *all* geometric figures in our games. I got busy.

Definition [S3]. Given figures f and F ; it is convenient to call a figure f a *F-clone* if f is homothetic to F .

Definition [S3]. Given a figure F . Let $S(F)$ be the minimum real number such that any finite set of F -clones of the combined area 1 can be packed in an F -clone of area $S(F)$.

Theorems 11 and 13-14 can be written in these notations as follows:

For a square \square , $S(\square) = 2$;

For any triangle Δ , $S(\Delta) = 2$.

However, it is easy to see that numbers $S(F)$ are not even bounded if we impose no limitation on figures F in the study:

Result 16 (A. Soifer, 1998, [S3]). For any number r , there is a figure F such that $S(F) > r$.

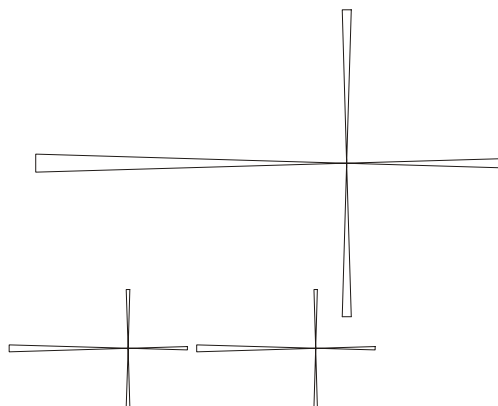


Figure 3

Proof. Indeed, for any r , we can construct a cross C thin enough so that only one of the two C -clones of area $\frac{1}{2}$ can be inscribed in a C -clone of area r (Figure 3).

Thus, it makes sense to limit the scope of our games to *convex* figures. The main problem then can be formulated as follows:

Main Open Problem 17 (A. Soifer, 1998, [S3]). For any convex figure F , find $S(F)$.

This is a difficult problem that in full generality may withstand centuries. However, partial solutions are possible and welcome.

I hope you have enjoyed the ride. Out of the window of our train of mathematical thought you may have noticed the terrain that has been continuously changing, with one problem giving birth to another. The journey is not over: in fact, it has only begun. We got a sense of packing some particular shapes, and are now ready to commence a search for its essence, a result encompassing all convex figures. In 1998 I came up with a bold conjecture:

Clones in Convex Figures Conjecture 18 (A. Soifer, 1998 [S3]). For *any* convex figure F , any set of F -clones F_1, F_2, \dots, F_n whose areas add up to at most 1, can be packed in a clone F_0 of area 2, i.e.,

$$S(F) \leq 2.$$

However, when I wrote this conjecture up for *Geombinatorics* [S3], I inadvertently put it as $S(F) = 2$. As the Russian proverb has it, *There is no bad without some good in it!* Three years later, in 2001, the Slovak geometer Pavel Novotný constructed a counterexample to the published equality $S(F) = 2$:

Novotný's Example 19 ([N], 2001). For rectangle R_0 of size $\sqrt{\frac{3}{2}} \times \sqrt{\frac{2}{3}}$ we get $S(R_0) = \sqrt{\frac{8}{3}} < 2$.

Novotný understood my typo, as he wrote "Soifer's conjecture could be changed to $S(F) \leq 2$ ".

A year later, in 2002, Janusz Januszewski [J1], beautifully completed the above result of Novotný:

Januszewski's Theorem 20 ([J1], 2002). For any rectangle R , $S(R) \leq 2$. Moreover, $S(R) = 2$ if and only if the rectangle R is a square.

Januszewski gave the main conjecture its final attribution:

The Soifer-Novotný Conjecture 21 ([J1], 2002). For any convex figure F , $S(F) \leq 2$.

Finally, Januszewski posed a natural problem that is a particular case of problem 17:

Januszewski's Problem 22 [J1]. Classify convex figures F for which $S(F) = 2$.

He among others noticed [J1] that, perhaps, the Soifer-Novotný Conjecture 21 can be generalized to n -dimensional Euclidean spaces:

n -Dimensional Conjecture 23. Let F be a convex body in an n -dimensional Euclidean space. Then any set of F -clones can be packed in an F -clone of volume $2^{n \delta 1}$.

I would also like to know the minimum value of $S(F)$:

Open Problem 24 Find $\min S(F)$ over all convex figures F and classify figures F for which this minimum is attained.

Most of these series of results appeared on pages of *Geombinatorics*, a quarterly dedicated to problem posing essays in combinatorial and discrete geometry (hence its title). Peter Winkler of Dartmouth University dedicated a section of his book [W, pp. 146 and 157] to the Discs in a Disc Conjecture:

This lovely conjecture is due to Alexander Soifer of the University of Colorado, Colorado Springs. It and its relatives have been the subject of a dozen of articles in the journal *Geombinatorics*; it is known, for example, that squares of total area 1 can be packed into a square of total area 2. The generalization to higher dimension was suggested by your author, among others; the case of two balls, each of volume $1/2$, shows that $2^{d \delta 1}$ is best possible.

I hope you have enjoyed your ride on this train of mathematical thought. However, the original 1932 conjecture is still open 82 years later, in the year 2014. It is time to double the prize:

CONJECTURE I. Hundred Dollar Squares in a Square Conjecture (P. Erdős). At perfect square numbers the function $f(r)$ does not increase: $f(k^2 + 1) = f(k^2)$.

CONJECTURE II. The Happy End Problem (Erdős-Szekeres 1933)

During the winter of 1932-33, two young friends, mathematics student Paul Erdős, aged 19, and chemistry student George (György) Szekeres, 21, solved the problem posed by their young lady friend Esther Klein, 22, but did not send it to a journal for a year and a half [ESz].

Erdős-Szekeres's Theorem 25 [ESz]. For any positive integer $n \geq 3$ there is an integer $ES(n)$

such that any set of at least $ES(n)$ points in the plane in general position³ contains n points that form a convex polygon.

The authors knew only two values, obtained by the members of their group of Jewish-Hungarian friends:

Esther Klein: $ES(4) = 5$.

E. Makai and Paul Turán independently: $ES(5) = 9$.

It is fascinating how sure Erdős and Szekeres were of their conjecture. In one of his last, posthumously published problem papers [E97.18], Erdős attached the prize and modestly attributed the conjecture to Szekeres: "I would certainly pay \$500 for a proof of Szekeres's conjecture."

CONJECTURE II: The Erdős-Szekeres Happy End \$500 Conjecture. $ES(n) = 2^{n-2} + 1$.

In 1933 Erdős and Szekeres proved lower and upper bounds for $ES(n)$; the conjectured value is their lower bound. The upper bound has only recently been improved first by Ronald L. Graham and Fan Chung and then by others, but is still pretty far from the conjectured value. Paul Erdős named it *The Happy End Problem*. He explained the name often in his talks. On June 4, 1992 in Kalamazoo I took notes of his talk:

I call it The Happy End Problem. Esther captured George, and they lived happily ever after in Australia. The poor things are even older than me.

This paper also convinced George Szekeres to become a mathematician. For Paul Erdős the paper had a happy end too: it became one of his early mathematical gems, Paul's first of the numerous contributions to and leadership of the Ramsey Theory and, as Szekeres put it, of "a new world of combinatorial set theory and combinatorial geometry."

The personages of The Happy End Problem appear to me like heroes of Shakespeare's plays. Paul, very much like *Tempest's* Prospero, gave up all his property, including books, to be free. George and Esther were so close, that they ended their lives together, like Romeo and Juliette. In the late summer 2005 e-mail, Tony Guttman conveyed to the world the sad news from Adelaide:

George and Esther Szekeres both died on Sunday morning [August 28, 2005]. George, 94, had been quite ill for the last 2-3 days, barely conscious, and died first. Esther, 95, died an hour later. George was one of the heroes of Australian mathematics, and, in her own way, Esther was one of the heroines.

On May 28, 2000, during a dinner in the restaurant of the Rydges North Sydney Hotel in Australia, George Szekeres told me "my student and I proved Esther's Conjecture for 17 with the use of computer," i.e., $ES(6) = 17$. "Which computer did you use?" asked I. "I don't care how pencil is made," answered George.

³ I.e., no three points lie on a line.

CONJECTURE III. Chromatic Number Conjecture (Soifer 2008)

In 1986, the Third Colorado Mathematical Olympiad included the following problem from our typical category: its solution is easy to see, especially when somebody has shown it to you.

3.2 Santa Claus and his elves paint the plane in two colors, red and green. Prove that the plane contains two points of the same color exactly one mile apart.

Solution. Toss on the plane an equilateral triangle with side lengths equal to one mile. Since its three vertices (pigeons) are painted in two colors (pigeonholes), there are two vertices painted in the same color (at least two pigeons in a hole). These two vertices *are* one mile apart.

You can prove the same result about a 3-colored plane:

No matter how the plane is 3-colored, it contains two points of the same color exactly one mile apart.

One may think that we know everything about the Euclidean plane. What else can there be after Pythagoras and Euclid, Steiner and Hilbert? First of all the Ancient Greeks did not think about these kinds of problems, where nothing is known about the coloring. Secondly, these simple problems are the starting points of a deep and still unresolved train of thought. For instance, try to push the problem to the next natural step, a 4-colored plane:

Is it true that no matter how the plane is 4-colored, it contains two points of the same color exactly one mile apart?

Imagine, nobody knows!

Chromatic Number of the Plane Problem 26. *What is the smallest number of colors with which we can color the plane in such a way that no color contains two points distance 1 apart?*

This number is called *the chromatic number of the plane* and is often denoted by χ . We can easily show (do) that 7 colors suffice, and thus $\chi < 7$. And so we know that $\chi = 4$ or 5 or 6 or 7. Which is it?

In August 1987 I attended an inspiring talk by the member of the U.S. National Academy of Sciences Paul Halmos at Chapman College in California. It was entitled "Some problems you can solve, some you cannot." This problem was an example of a problem "you cannot solve." So far Halmos is correct.

While writing *The Mathematical Coloring Book* [S5], in ca. 2007, I formulated the following conjecture:

Chromatic Number of the Plane Conjecture 27 (Soifer 2007). $\chi = 7$.

If you are familiar with 3- and generally n -dimensional Euclidean space E^n , you will readily see that this problem straight-forwardly generalizes to n dimensions, and we can ask a more general question of the chromatic number of E^n . In [S5] I have conjectured a simple formula for the chromatic number of the n -dimensional Euclidean space E^n :

CONJECTURE III: Chromatic Number of n -Space. $\chi(E^n) = 2^{n+1} \pm 1$.

As Paul Erdős used to say, "If true, this conjecture may take centuries to prove, but we shall see!"

CONJECTURE IV. Triangular Covering (Conway-Soifer 2004)

In 2004 we held the 21st Colorado Mathematical Olympiad, for which I created the following problem [S6]:

21.4. To Have a Cake

(A) We need to protect from the rain a cake that is in the shape of an equilateral triangle of side 2.1. All we have are identical tiles in the shape of an equilateral triangle of side 1. Find the smallest number of tiles needed.

(B) Suppose the cake is in the shape of an equilateral triangle of side 3.1. Will 11 tiles be enough to protect it from the rain?

Solution (A). Mark 6 points in the equilateral triangle of side 2.1: its vertices and midpoints of the sides (Fig. 4). A tile can cover at most one such point, therefore we need at least 6 tiles.

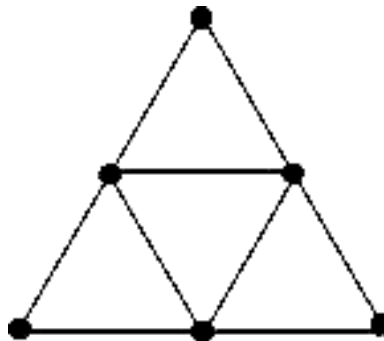


Figure 4

On the other hand, 6 tiles can do the job. There are different ways to achieve it. Here is one. We can first cover the three corners (Fig. 5 left), and then use 3 more tiles to cover the remaining hexagon (Fig. 5 right).

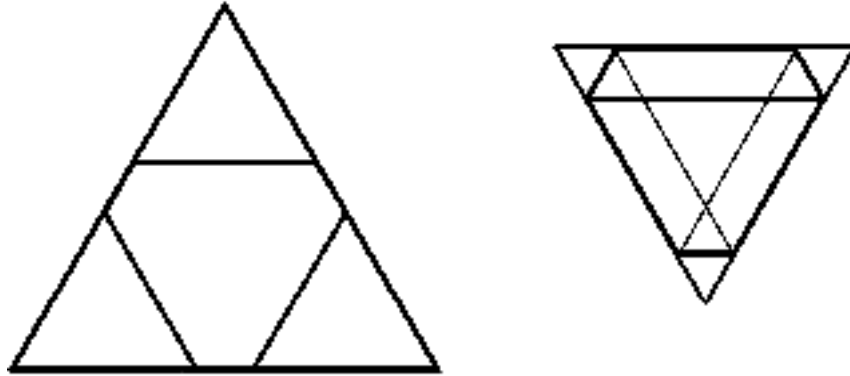


Figure 5

(B). We can use 4 tiles to cover the top triangle of side 2, and then use the remaining 7 tiles for a bottom trapezoid (Fig. 6).

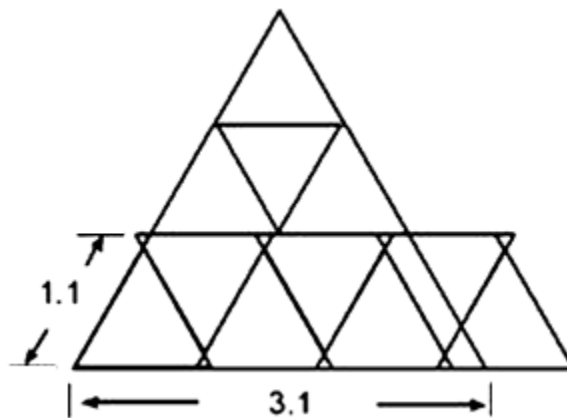


Figure 6

Have you noticed that I did not ask the Olympians to prove that 11 covering tiles are necessary? At the Olympiad, I could only ask what I can prove myself!

Upon my return to Princeton, where I worked at the time, I shared a more general form of this problem with John H. Conway. Imagine, we both found proofs of the sufficient condition, which were markedly different. And so John and I decided to set a world record: to publish an article containing just one word in its text. Let me reproduce here our submission to *The American Mathematical Monthly*.

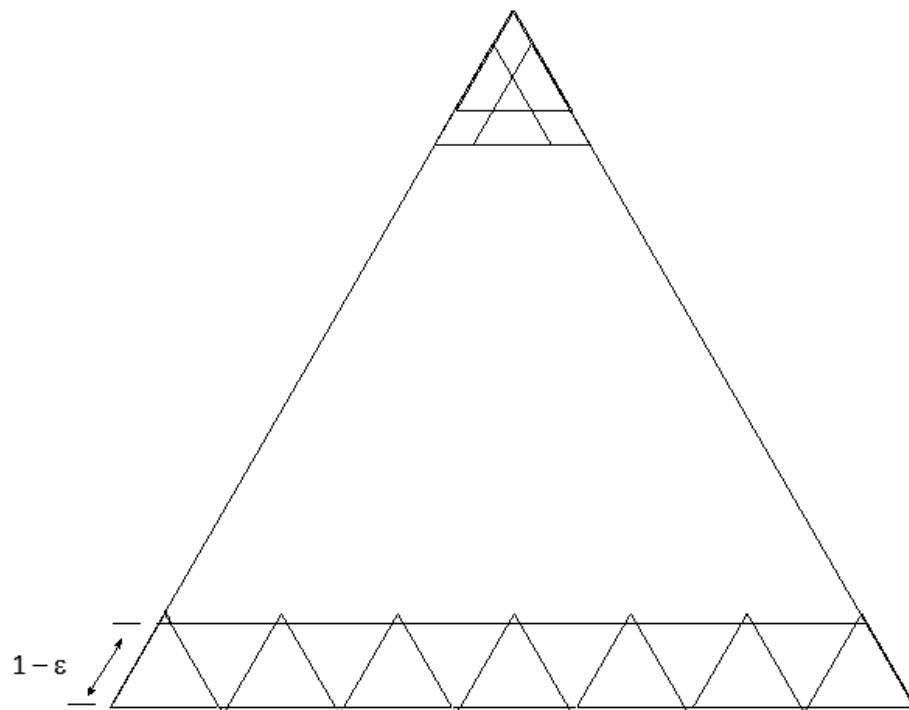
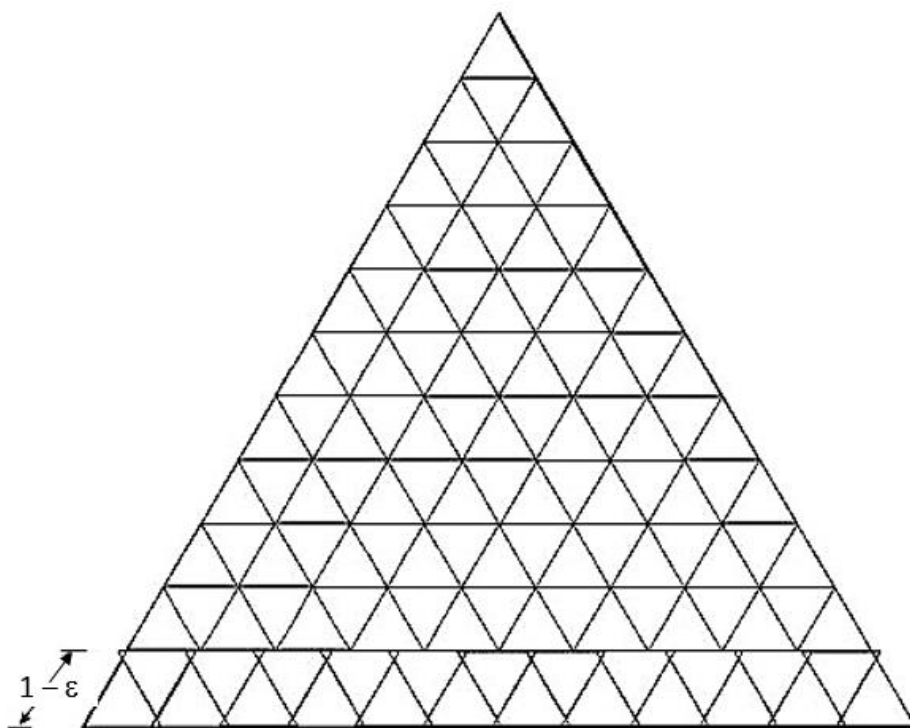
Can $n^2 + 1$ unit equilateral triangles cover an equilateral triangle of side $> n$, say $n + \varepsilon$?

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$n^2 + 2$ can:



The American Mathematical Monthly was puzzled. On April 30, 2004, Editorial Assistant of the Monthly Margaret A. Combs sent me an e-mail:

The Monthly publishes exposition of mathematics at many levels, and it contains articles both long and short. Your article, however, is a bit too short to be a good Monthly article ... A line or two of explanation would really help.

Having learned from me about *The Monthly* reply, John Conway exclaimed, "Do not give up too easily!" And so, I replied the same day as follows:

I respectfully disagree that a short paper in general and this paper in particular are merely due to its size must be "a bit too short to be a good Monthly article." Is there a connection between quantity and quality? ... We have posed a fine (in our opinion) open problem and reported two distinct "behold-style" proofs of our advance on this problem. What else is there to explain?

The Monthly published our article [CS], but spoiled our single-word world record by unilaterally including our title in the body of the article!

John Conway believed that since his and my coverings were so vastly distinct, the problem was too hard to continue fighting with it. However, shortly after, the Columbia University undergraduate student Dmytro Karabash joined me in working on this problem. We generalized the problem to covering an arbitrary triangle T . Tiling triangles will be similar to T and their corresponding sides will be $n + \varepsilon$ times smaller and let us call them $1/(n + \varepsilon)$ -clones of T .

Result 28 (Karabash-Soifer, 2005) Any non-equilateral triangle T can be covered by $n^2 + 1$ $1/(n + \varepsilon)$ -clones of T .

Proof: An appropriate affine transformation maps equilateral triangle on Figure 9 onto T . This transformation gives a covering of T with $n^2 + 2$ tiling clones, but now we can cover the transformed top triangle (see Figure 9) with 2 clones instead of 3 as shown in Figure 10, thus reducing the total number of covering clones to $n^2 + 1$.

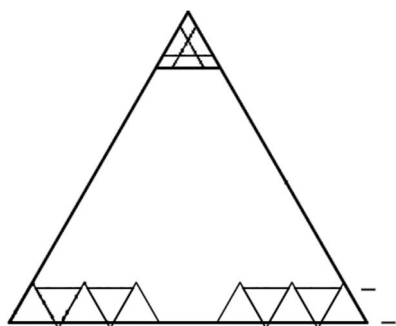


Figure 9

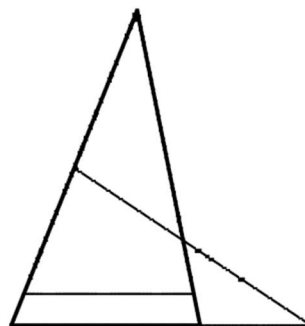


Figure 10

Dmytro and I also generalized the problem by introducing *trigons* [KS]. But we were unable to prove the Conway-Soifer Conjecture that equilateral triangle requires $n^2 + 2$ covering triangles.

Imagine, the equilateral triangle proved to be the hardest of all! You have a chance to prove it yourselves ó sharpen your pencils!

TRIANGULAR COVERING CONJECTURE IV (Conway-Soifer 2004). An equilateral triangle of side n cannot be covered by $n^2 + 1$ equilateral triangles of side $1 \pm \varepsilon$.

The smallest open case is the following conjecture:

The Hexagon Conjecture (Karabash-Soifer 2005). Seven equilateral triangles of side $1 \pm \varepsilon$ cannot cover an equilateral hexagon of side 1.

Acknowledgment. I thank Robert Ewell for cleaning up my illustrations.

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